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The Feynman Nonrelativistic Chessboard

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The sum-over-paths prescription of the Feynman chessboard model is adapted directly to the nonrelativistic case. The simple binary geometry is sufficient to obtain the usual Feynman propagator for the free particle in the continuum limit.

1. INTRODUCTION

The relativistic chessboard (Feynman and Hibbs, 1965; Gersch, 1981; Jacobson and Schulman, 1984; Ord, 1992) was an attempt by Feynman to generalize his space-time approach to include special relativity. The attempt was incomplete, although some recent work suggests that the chessboard model may prove more fundamental than the better known nonrelativistic path integral (Ord, 1992, 1993; Ord and McKeon, 1993).

In this note the "chessboard prescription" for the sum over paths is translated into the nonrelativistic domain and the nonrelativistic propagator obtained as a result.

2. THE CHESSBOARD MODEL

In the Feynman chessboard model a particle is constrained to move on a space-time lattice with spacings δ and ϵ in x and t, respectively. At each step in discrete time the particle moves either one lattice spacing to the left or right. The kernel $K_\epsilon(b, a)$ for a particle to propagate from position a at time t_a to position b at time t_b is given by

$$K_\epsilon(b, a) = \sum_R N(R)(i\epsilon m)^R \tag{1}$$

where the sum is over all paths partioned according to the number of corners

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(direction changes) R. Here $N(R)$ is the number of paths with R corners connecting a and b.

In the chessboard model, the lattice is refined in such a way that

$$\lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \frac{\delta}{\epsilon} = c \tag{2}$$

As shown by Jacobson and Schulman (1984), this limit results in paths which are ultimately piecewise linear with an expected time of $1/m$ between direction changes.

To adapt this prescription to the nonrelativistic case we have to remove the relativistic scaling (2) and replace it with diffusive scaling, namely

$$\lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \frac{\delta^2}{2\epsilon} = \frac{1}{2m} \qquad (\hbar = 1) \tag{3}$$

This scaling removes the mean free time $1/m$ between collisions and builds in the uncertainty principle on all scales.

The nonrelativistic prescription is then

$$K_\epsilon(b, a) = \sum_R N(R)(i)^R 2^{-t/2\epsilon} \tag{4}$$

where $N(R)$ is the number of R-cornered paths between a and b. Here $t = t_b - t_a$ and we have $\delta^2/2\epsilon = 1/(2m)$. One further restriction we shall require is that lattice refinements be carried out in such a way that $N = t/\epsilon = 0$ (mod 8). The reason for this will soon be apparent.

As in the relativistic case the propagator (4) is a 2×2 matrix with elements indexed by the arrival and departure directions on the lattice. One way to obtain $K_\epsilon(b, a)$ is to consider the difference equation that must be satisfied by the two-component amplitudes of the system. If $\phi_\pm(x, t)$ is the amplitude for the particle to be at $x = m\delta$ at time $t = N\epsilon$ moving in the $+$ $(-)$ direction, then we have

$$\begin{aligned} \phi_+(m\delta, (N+1)\epsilon) &= (\phi_+((m-1)\delta, N\epsilon) + i\phi_-(m\delta, N\epsilon))/\sqrt{2} \\ \phi_-(m\delta, (N+1)\epsilon) &= (i\phi_+((m\delta, N\epsilon) + \phi_-((m+1)\delta, N\epsilon))/\sqrt{2} \end{aligned} \tag{5}$$

Now write

$$\Phi_\pm(p, N\epsilon) = \sum_{-\infty}^{+\infty} e^{-ipm\delta} \phi_\pm(m\delta, N\epsilon)\delta \tag{6}$$

Multiply the left-hand side of (5) by $e^{-ipm\delta}$ and sum to get

$$\begin{pmatrix}\phi_+(p,(N+1)\epsilon)\\ \phi_-(p,(N+1)\epsilon)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-ip\delta} & i\\ i & e^{ip\delta}\end{pmatrix}\begin{pmatrix}\phi_+(p,N\epsilon))\\ \phi_-(p,N\epsilon))\end{pmatrix}$$

$$= T^{N+1}\begin{pmatrix}\phi_+(p,0)\\ \phi(p,0)\end{pmatrix} \tag{7}$$

where T is the transfer matrix

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{ip\delta} & i\\ i & e^{ip\delta}\end{pmatrix} \tag{8}$$

This has eigenvalues

$$\lambda_{\pm} = \frac{\cos p\delta}{\sqrt{2}} \pm \frac{i}{2}(3 - \cos 2p\delta)^{1/2}$$

$$= e^{\pm\pi/4}\left(1 \pm \frac{ip^2\delta^2}{2}\right) + O(\delta^4) \tag{9}$$

In the continuum limit we shall want to find $\lambda_{\pm}^{N}$, where $N = t/\epsilon = t/m\delta^2$, assuming that N is $O(\text{mod } 8)$. In this limit

$$\lambda_{\pm} \to e^{\pm ip^2t/2m}$$

and the kernel in (7) becomes

$$k(p,t) = \begin{bmatrix}\cos(p^2t/2m) & i\sin(p^2t/2m)\\ i\sin(p^2t/2m) & \cos(p^2t/2m)\end{bmatrix} \tag{10}$$

If we diagonalize (10), we get

$$K(p,t) = \begin{bmatrix}e^{ip^2t/2m} & 0\\ 0 & e^{-ip^2t/2m}\end{bmatrix} \tag{11}$$

and it is clear that each component of the amplitude associated with (11) satisfies a Schrödinger equation. The two-component form of (11) suggests that the amplitudes are the one-dimensional analogs of spinors.

There are a few things to note about the above result.

(a) The form of the continuum propagator is a result of the simple binary geometry of the paths. At each lattice refinement, all paths have exactly two choices of direction at each step. This can be thought of as an inheritance from the relativistic case. There it was necessary to ensure that the eigenstates of the velocity operator had eigenvalues $\pm c$. Here it appears necessary (or at least sufficient) to provide paths with sufficient direction correlations that period-four correlations ($i^4 = 1$) produce the Feynman propagator.

(b) The configuration space path integral associated with this formulation is, in a sense, slightly more honest than its predecessors. In this case we have started off on a lattice with a genuine lattice propagator [equation (4)] and proceeded to obtain the resulting continuum object. Typical derivations of the path integral (Feynman and Hibbs, 1965) start with a continuum short-time propagator and proceed to produce the long-time propagator by placing the short-time propagator on a lattice and taking a limit. This procedure clearly works, although, as will be shown elsewhere, assumption of a continuum propagator at the outset constitutes a projection from an underlying lattice model. This projection hides some interesting features of the transition from lattice to continuum.

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